

# Laplace-Beltrami operator and exact solutions for branes

A. A. Zheltukhin <sup>a,b\*</sup>

<sup>a</sup> Kharkov Institute of Physics and Technology,  
1, Akademicheskaya St., Kharkov, 61108, Ukraine

<sup>b</sup> Nordita

Royal Institute of Technology and Stockholm University  
Roslagstullsbacken 23, SE-106 91 Stockholm, Sweden

## Abstract

Proposed is a new approach to finding exact solutions of nonlinear  $p$ -brane equations in  $D$ -dimensional Minkowski space based on the use of various initial value constraints. It is shown that the constraints  $\Delta^{(p)}\vec{x} = 0$  and  $\Delta^{(p)}\vec{x} = -\Lambda(t, \sigma^r)\vec{x}$  give two sets of exact solutions.

## 1 Introduction

Branes are fundamental constituents of string theory [1], but not so much is known about their internal structure encoded in the nonlinear PDEs [2-14]. As a result, the emphasis in the investigations is shifted to exploring various particular solutions and the physics based on them. There is a progress in search for spinning membranes ( $p = 2$ ) with spherical/toroidal topology embedded in flat and curved  $AdS_p \times S^q$  backgrounds (see e.g. [15-20]). Extension of these results to the case  $p = 3$  and complexified backgrounds with symmetry groups such as  $SU(n) \times SU(m) \times SU(k)$  was done in [20], where radial stability of three-branes was established. Analysis of spinning branes

---

\*e-mail: aaz@physto.se

with higher  $p$ , as well as finding other particular solutions of brane equations is an open problem. On this way an important observation was done by Hoppe in [21], where the  $U(1)$ -invariant ansatz reducing the membrane equations in  $D = 5$  Minkowski space to the system of two-dimensional nonlinear equations was proposed. The particular solutions of the Hoppe equations which describe collapsing or spinning flat tori in  $D = 5$  were found in [22] and their connection with the geometric approach, Abel and pendulum differential equations was established in [23]. The extension of the membrane ansatz describing the Abelian  $U(1)^p$  invariant  $p$ -branes revealed exact hyperelliptic solutions for flat  $p$ -tori embedded into  $D = (2p + 1)$ -dimensional Minkowski space [24]. Exact solutions corresponding to spinning  $p$ -branes in  $D = (2p + 1)$ -dimensional Minkowski space were found in [25].

Here we make an attempt to understand the above-mentioned exact  $p$ -brane solutions on the base of a general approach which allows to find new exact solutions. The approach uses the wave representation of  $p$ -brane equations on the  $(p + 1)$ -dimensional worldvolume  $\Sigma_{p+1}$ . In the orthogonal gauge these wave equations are reduced to the ones including Laplace-Beltrami operator  $\Delta^{(p)}$  on the hypersurface  $\Sigma_p$ . We propose to classify the brane solutions exploring various initial value constraints imposed on  $\Delta^{(p)}\vec{x}$ . We show that the harmonicity constraints  $\Delta^{(p)}\vec{x} = 0$  pick up the solutions describing spinning  $p$ -branes which include the spinning ansatz [25] in the case  $D = 2p + 1$ . These solutions include the infinite  $p$ -branes with the shape of hyperplanes which are reduced to  $p$ -dimensional domain walls with the constant brane energy density in the static limit. Found also are periodic solutions describing closed spinning folded  $p$ -branes with a singular metric, which generalize the folded string solutions [26], [27] to the case of  $p$ -branes. The effect of the formation of singularities for closed strings and membranes was also discussed in [28].

Further, the present paper reveals that the harmonicity constraints  $\Delta^{(p)}\vec{x} = -\Lambda(t, \sigma^r)\vec{x}$  select the exact solutions with  $\Lambda = \frac{p}{\bar{R}^2(t)}$  describing closed  $p$ -branes with their hypersurface  $\Sigma_p$  lying on the collapsing sphere  $S^{D-2}$  with the time-dependent radius equal to  $\sqrt{\bar{R}^2}$ . The nonlinear equation for  $R(t)$  turns out to be exactly solvable for any dimension  $D$  of the Minkowski space and results in hyperelliptic functions. In the case  $D = 2p + 1$  these solutions are reduced to the degenerate ansatz [24] with all equal radii of the corresponding  $p$ -tori. The presence of such collapsing solutions generated by the deformed harmonicity constraint is a common property of closed membranes

and  $p$ -branes independent of the Minkowski space dimension  $D \geq p + 1$ .

## 2 Worldvolume wave equations for branes

The Dirac action for a  $p$ -brane without boundaries is defined by the integral

$$S = T \int \sqrt{|G|} d^{p+1} \xi, \quad (1)$$

in the dimensionless worldvolume parameters  $\xi^\alpha$  ( $\alpha = 0, \dots, p$ ). The components  $x^m = (t, \vec{x})$  of the brane world vector in the  $D$ -dimensional Minkowski space with the signature  $\eta_{mn} = (+, -, \dots, -)$  have the dimension of length, and the dimension of tension  $T$  is  $L^{-(p+1)}$ . The induced metric  $G_{\alpha\beta} := \partial_\alpha x_m \partial_\beta x^m$  is presented in  $S$  by its determinant  $G$ .

After splitting the parameters  $\xi^\alpha := (\tau, \sigma^r)$  the Euler-Lagrange equations and  $(p + 1)$  primary constraints generated by  $S$  take the form

$$\partial_\tau \mathcal{P}^m = -T \partial_r (\sqrt{|G|} G^{r\alpha} \partial_\alpha x^m), \quad \mathcal{P}^m = T \sqrt{|G|} G^{\tau\beta} \partial_\beta x^m, \quad (2)$$

$$\tilde{T}_r := \mathcal{P}^m \partial_r x_m \approx 0, \quad \tilde{U} := \mathcal{P}^m \mathcal{P}_m - T^2 |\det G_{rs}| \approx 0, \quad (3)$$

where  $\mathcal{P}^m$  is the energy-momentum density of the brane.

It is convenient to use the orthogonal gauge simplifying the metric  $G_{\alpha\beta}$

$$\begin{aligned} L\tau = x^0 \equiv t, \quad G_{\tau\tau} = -L(\dot{\vec{x}} \cdot \partial_r \vec{x}) = 0, \\ g_{rs} := \partial_r \vec{x} \cdot \partial_s \vec{x}, \quad G_{\alpha\beta} = \begin{pmatrix} L^2(1 - \dot{\vec{x}}^2) & 0 \\ 0 & -g_{rs} \end{pmatrix} \end{aligned} \quad (4)$$

with  $\dot{\vec{x}} := \partial_t \vec{x} = L^{-1} \partial_\tau \vec{x}$ . As a result, the constraint  $\tilde{U}$  (3) represents  $\mathcal{P}_0$  as

$$\mathcal{P}_0 = \sqrt{\vec{\mathcal{P}}^2 + T^2 |g|}, \quad g = \det(g_{rs}) \quad (5)$$

and it becomes the Hamiltonian density  $\mathcal{H}_0$  of the  $p$ -brane since  $\dot{\mathcal{P}}_0 = 0$  in view of Eq.(2). Using the definition of  $\mathcal{P}_0$  (2) and  $G^{\tau\tau} = 1/L^2(1 - \dot{\vec{x}}^2) = 1/L^2 G^{tt}$  we express  $\mathcal{P}_0$  as a function of the velocity  $\dot{\vec{x}}$

$$\mathcal{P}_0 := TL \sqrt{|\det G|} G^{\tau\tau} = T \sqrt{\frac{|g|}{1 - \dot{\vec{x}}^2}}. \quad (6)$$

Taking into account this expression and definition (2) one can present  $\vec{\mathcal{P}}$  and its evolution equation (2) in the form previously used in [28], [22] and [24],

$$\vec{\mathcal{P}} = \mathcal{P}_0 \dot{\vec{x}}, \quad \dot{\vec{\mathcal{P}}} = T^2 \partial_r \left( \frac{|g|}{\mathcal{P}_0} g^{rs} \partial_s \vec{x} \right). \quad (7)$$

Then Eqs. (7) produce the second-order PDE for  $\vec{x}$

$$\ddot{\vec{x}} = \frac{T}{\mathcal{P}_0} \partial_r \left( \frac{T}{\mathcal{P}_0} |g| g^{rs} \partial_s \vec{x} \right). \quad (8)$$

These equations may be presented in the canonical Hamiltonian form

$$\dot{\vec{x}} = \{H_0, \vec{x}\}, \quad \dot{\vec{\mathcal{P}}} = \{H_0, \vec{\mathcal{P}}\}, \quad \{\mathcal{P}_i(\sigma), x_j(\tilde{\sigma})\} = \delta_{ij} \delta^{(p)}(\sigma^r - \tilde{\sigma}^r),$$

where  $H_0$  is the integrated Hamiltonian density  $\mathcal{H}_0 \equiv \mathcal{P}_0$

$$H_0 = \int d^p \sigma \sqrt{\vec{\mathcal{P}}^2 + T^2 |g|}. \quad (9)$$

The presence of square root in (9) points to the presence of the known residual symmetry preserving the orthogonal gauge (4)

$$\tilde{t} = t, \quad \tilde{\sigma}^r = f^r(\sigma^s) \quad (10)$$

and generated by the constraints  $\tilde{T}_r$  (3) reduced to the form

$$T_r := \vec{\mathcal{P}} \partial_r \vec{x} = 0 \quad \Leftrightarrow \quad \dot{\vec{x}} \partial_r \vec{x} = 0, \quad (r = 1, 2, \dots, p). \quad (11)$$

The freedom allows to impose  $p$  additional time-independent conditions on  $\vec{x}$  and its space-like derivatives. The presented description does not restrict space-time and brane worldvolume dimensions  $(D, p)$  and  $p < D$ .

Alternatively, we present  $p$ -brane Eqs. (2) as the reparametrization invariant wave equation for  $x^m$  on the  $(p+1)$ -dim. brane worldvolume  $\Sigma_{p+1}$

$$\square^{(p+1)} x^m = 0, \quad (12)$$

where  $\square^{(p+1)} := \frac{1}{\sqrt{|G|}} \partial_\alpha \sqrt{|G|} G^{\alpha\beta} \partial_\beta$  is the Laplace-Beltrami operator.

Using the relation  $\partial_\alpha \ln \sqrt{|G|} = \Gamma_{\alpha\beta}^\beta$ , where  $\Gamma_{\alpha\beta}^\gamma$  are the Cristoffel symbols generated by the metric  $G_{\alpha\beta}$  of  $\Sigma_{p+1}$ , one can express Eqs.(12) as the vanishing covariant divergence of the worldvolume vector  $x^{m,\alpha}$

$$\square^{(p+1)} x^m \equiv \nabla_\alpha x^{m,\alpha} = 0, \quad (13)$$

where  $x^{m,\alpha} := G^{\alpha\beta}\partial_\beta x^m$  and  $\nabla_\alpha x^{m,\alpha} \equiv \partial_\alpha x^{m,\alpha} + \Gamma_{\beta\alpha}^\alpha x^{m,\beta}$ . Eqs.(13) are presented as the continuity equations

$$\partial_\alpha T^{m\alpha} = 0$$

for the components of Noether current  $T^{m\alpha} := T\sqrt{|G|}G^{\alpha\beta}\partial_\beta x^m$  generated by the global translation symmetry of the Minkowski target space.

Below, we shall use the wave representation (12) in a fixed gauge to develop a way for construction of some exact solutions of the brane equations.

### 3 Laplace-Beltrami operator and Noether identities for p-branes

Using the gauge (4) one can extract the Laplace-Beltrami operator  $\Delta^{(p)}$ , associated with the p-brane hypersurface  $\Sigma_p$ , from the operator  $\square^{(p+1)}$

$$\Delta^{(p)}\vec{x} := \frac{1}{\sqrt{|g|}}\partial_r \left( \sqrt{|g|}g^{rs}\partial_s \vec{x} \right), \quad (14)$$

where  $g_{rs} := \partial_r \vec{x} \cdot \partial_s \vec{x}$  is the induced metric on  $\Sigma_p$ . The use of the LB operator  $\Delta^{(p)}$  allows to present Eqs. (12) as the system of  $(D-1)$  equations

$$\ddot{\vec{x}} = \frac{1}{2}\partial_r(1 - \dot{\vec{x}}^2)g^{rs}\partial_s \vec{x} + (1 - \dot{\vec{x}}^2)\Delta^{(p)}\vec{x}. \quad (15)$$

Taking into account the relation  $\frac{1}{2}\partial_r(1 - \dot{\vec{x}}^2) = (\ddot{\vec{x}}\partial_r \vec{x})$ , following from the orthogonality conditions (11), we rewrite the system (15) in the form

$$\ddot{\vec{x}} - (\ddot{\vec{x}}\vec{x}^r)\vec{x}_{,r} = (1 - \dot{\vec{x}}^2)\Delta^{(p)}\vec{x}, \quad (16)$$

where the following condensed notations are used

$$\vec{x}_{,r} := \partial_r \vec{x}, \quad \vec{x}^{,r} := g^{rs}\vec{x}_{,s} \rightarrow \vec{x}_{,r}\vec{x}^{,s} = \delta_r^s. \quad (17)$$

Eqs. (16) show equality between two invariants of the residual diffeomorphisms (10) of  $\Sigma_p$  one of which is  $\Delta^{(p)}\vec{x}$ , including only the space-like derivatives of  $\vec{x}$ , and the other

$$I := G^{tt}[\ddot{\vec{x}} - (\ddot{\vec{x}}\vec{x}^r)\vec{x}_{,r}] \quad (18)$$

capturing all time-like derivatives of  $\vec{x}$ .  $I$  equals the metric component  $G^{tt} = 1/(1 - \dot{\vec{x}}^2)$  multiplied by the l.h.s. of Eqs.(16) equal to projection of the acceleration  $\ddot{\vec{x}}$  on the directions orthogonal to  $\Sigma_p$ . This follows from the identities

$$\vec{x}_{,r}[\ddot{\vec{x}} - (\ddot{\vec{x}}\vec{x}^s)\vec{x}_{,s}] \equiv 0 \quad (19)$$

that imply that  $(\vec{x}_{,r}\Delta^{(p)}\vec{x}) \equiv 0$  which are a consequence of the formula

$$\Delta^{(p)}\vec{x} \equiv \nabla_s \vec{x}^s = \partial_s \vec{x}^s + (\partial_s \ln \sqrt{|g|})\vec{x}^s. \quad (20)$$

The covariant derivative  $\nabla_r \vec{x}^s := \partial_r \vec{x}^s + \Gamma_{rq}^s \vec{x}^q$  contains the Cristoffel symbols  $\Gamma_{ps}^r$  constructed from the metric tensor  $g_{rs}$  of the brane hypersurface  $\Sigma_p$ . Indeed, the representation (20) multiplied by the vectors  $\vec{x}_{,r}$  results in

$$(\vec{x}_{,r}\Delta^{(p)}\vec{x}) = \partial_r \ln \sqrt{|g|} + \vec{x}_{,r} \partial_s \vec{x}^s = \partial_r \ln \sqrt{|g|} - \frac{1}{2}(g^{sq} \partial_r g_{qs}) \equiv 0 \quad (21)$$

in view of the well-known relation  $g^{sq} dg_{qs} = d \ln |g|$ . The derived identities (19) extracted from Eqs.(16) are the Noether identities associated with the residual gauge symmetry (10) of the  $p$ -brane equations.

From the physical point of view the brane Eqs.(16) mean that the constituent of  $\ddot{\vec{x}}$  orthogonal to  $\Sigma_p$  is parallel to  $\Delta^{(p)}\vec{x}$ , and therefore the forces orthogonal to the brane hypersurface are represented by the vector  $\Delta^{(p)}\vec{x}$ . The geometric interpretation of the invariant  $I$  allows to express the brane equations (16) in the equivalent form

$$\Pi_{ik} \ddot{x}_k = (1 - \dot{\vec{x}}^2) \Delta^{(p)} x_k, \quad (22)$$

where  $\Pi_{ik}$  is the projection operator

$$\Pi_{ik} := \delta_{ik} - x_{i,r} x_k^r, \quad \Pi_{ik} \Pi_{kl} = \Pi_{il} \quad (23)$$

on the local vectors  $\vec{n}_\perp$  orthogonal to the tangent vectors  $\vec{x}_{,r}$  of  $\Sigma_p$ . Then the property of orthogonality of  $\Delta^{(p)}\vec{x}$  to  $\Sigma_p$  is encoded by the conditions

$$\Pi_{ik} \Delta^{(p)} x_k = \Delta^{(p)} x_i \quad (24)$$

showing that  $\Delta^{(p)}\vec{x}$  is an eigenvector of the projection operator  $\Pi_{ik}$  similarly to the Euclidean vectors  $\vec{n}_\perp$  and  $\dot{\vec{x}}$

$$\Pi_{ik} \dot{x}_k = \dot{x}_i, \quad \Pi_{ik} n_{\perp k} = n_{\perp i}. \quad (25)$$

The presence of  $p$  Noether identities (19) proves that  $(D - 1)$  brane equations (16) contain only  $(D - p - 1)$  independent equations

$$\vec{n}_\perp[\ddot{\vec{x}} - (1 - \dot{\vec{x}}^2)g^{rs}\vec{x}_{,rs}] = 0, \quad (26)$$

generated by the projections of (16) on the vectors  $\vec{n}_\perp(t, \sigma^r)$  orthogonal to the tangent hyperplane spanned by the vectors  $\partial_r \vec{x}$  at the point  $(t, \sigma^r)$

$$\vec{n}_\perp \vec{x}_{,s} = 0 \quad \rightarrow \quad \vec{n}_\perp \partial_r \vec{x}^{,r} = g^{rs}(\vec{n}_\perp \vec{x}_{,rs}), \quad (27)$$

where the subindex  $\perp = p + 1, p + 3, \dots, D - 1$  takes  $(D - p - 1)$  values.

Using  $G_{\alpha\beta}$  (4) one can present Eqs.(26) in an equivalent form

$$G^{\alpha\beta} W_{\alpha\beta}^\perp \equiv G^{\alpha\beta}(\vec{n}_\perp \vec{x}_{,\alpha\beta}) = 0 \quad (28)$$

recognized as the minimality conditions for the worldvolume  $\Sigma_{p+1}$  embedded in the  $D$ -dimensional Minkowski space expressed via the covariant traces of the second fundamental form  $W_{\alpha\beta}^\perp$  of the brane worldvolume  $\Sigma_{p+1}$ .

In the considered orthogonal gauge (4) the  $(p + 1)$ -st Noether identity, associated with the freedom in  $\tau$ -reparametrizations of  $\Sigma_{p+1}$ , reduces to the energy density conservation  $\dot{\mathcal{P}}_0 = 0$ . It can be seen when analyzing the projection of (16) on the vector  $\dot{\vec{x}}$ . Really, taking into account the relations

$$\dot{\vec{x}} \Delta^{(p)} \vec{x} = -\frac{1}{2}(g^{sq} \partial_t g_{qs}) = -\partial_t \ln \sqrt{|g|}, \quad (29)$$

$$\dot{\vec{x}} \frac{\ddot{\vec{x}} - (\ddot{\vec{x}} \dot{\vec{x}}^s) \dot{\vec{x}}_{,s}}{1 - \dot{\vec{x}}^2} = -\partial_t \ln \sqrt{1 - \dot{\vec{x}}^2}. \quad (30)$$

one can present the projection of Eqs.(16) on  $\dot{\vec{x}}$  as

$$\partial_t \ln \sqrt{1 - \dot{\vec{x}}^2} = \partial_t \ln \sqrt{|g|} \quad (31)$$

or, after using definition (6) for the energy density  $\mathcal{P}_0$ , in the form

$$\partial_t \ln \sqrt{\frac{|g|}{1 - \dot{\vec{x}}^2}} = \partial_t \ln \left( \frac{\mathcal{P}_0}{T} \right) = 0. \quad (32)$$

Eq. (32) is satisfied in view of the above-proved energy conservation law.

## 4 Solvable $p$ -brane motions with $\Delta^{(p)}\vec{x} = 0$

The interpretation of  $\Delta^{(p)}\vec{x}$  as the vector encoding forces orthogonal to  $\Sigma_p$  may be used for exploring admissible motions of branes. On this way it is natural to study the motions in the absence of forces orthogonal to the brane hypersurface  $\Sigma_p$ . These motions are fixed by the harmonicity conditions

$$\Delta^{(p)}\vec{x} = 0 \quad (33)$$

which must be considered as the initial value constraints for brane Eqs.(16). Since the constraints (33) have to be preserved in time the corresponding brane evolution must obey the following equations

$$\ddot{\vec{x}} - (\ddot{\vec{x}}\vec{x}^r)\vec{x}_{,r} = 0, \quad (34)$$

as it follows from Eqs.(24). It is easy to see that Eqs. (34) have a particular solution that coincides with the general solution of the system

$$\ddot{\vec{x}} = 0, \quad \Delta^{(p)}\vec{x} = 0 \quad (35)$$

which describes the motions in the balance of forces acting on the brane. The general solution of evolution Eqs. (35) is linear in time

$$\vec{x} = \vec{x}_0(\sigma^r) + \vec{v}_0(\sigma^r)t, \quad \vec{v}_0\vec{x}_{0,r} = 0, \quad \vec{v}_0^2 = \text{constant}, \quad (36)$$

as it follows from the orthogonality conditions (11). Then harmonicity conditions (35) are transformed to constraints for the initial values  $\vec{x}_0(\sigma^r)$  and  $\vec{v}_0(\sigma^r)$ . The static  $p$ -branes are described by the particular solution

$$\vec{x} = \vec{x}_0(\sigma^r), \quad \Delta^{(p)}\vec{x}_0 = 0 \quad (37)$$

and the harmonicity conditions yield the initial data constraints for the brane shape  $\vec{x}_0(\sigma^r)$ . The static brane energy density  $\mathcal{P}_0^{(stat)} = T\sqrt{|g|}$  and it can realize the ground state of  $p$ -brane, as its kinetic energy vanishes. Let us note that an antipode of the static brane is the one moving with the maximum velocity equals the velocity of light, i.e.  $\dot{\vec{x}}^2 = 1$ . In this case Eqs. (16) are reduced to the above-discussed equation  $\ddot{\vec{x}} = 0$ , but with arbitrary  $\Delta^{(p)}\vec{x}$ . The branes moving with the velocity of light have zero tension and degenerate metric (4) of their worldvolumes [8]. The discussed examples of particular solutions confirm correctness of the proposed approach for exploring solutions



of Eqs. (22). So, one can apply it for studying the general solution of (34) describing tensionfull branes characterized by  $(1 - \dot{\vec{x}}^2) > 0$ .

Generally Eqs.(34) capture the whole set of motions characterized by zero projections of the acceleration  $\ddot{\vec{x}}$  on the directions orthogonal to  $\Sigma_p$

$$\Delta^{(p)}\vec{x} = 0 = \ddot{\vec{x}} - (\ddot{\vec{x}}\vec{x}^r)\vec{x}_{,r} \longrightarrow \ddot{\vec{x}}\vec{x} = 0 \longrightarrow \dot{\vec{x}}^2 = \vec{v}^2(\sigma^r). \quad (38)$$

The forces acting on the brane are tangent to  $\Sigma_p$  and produce acceleration orthogonal to the velocity  $\dot{\vec{x}}$ , respectively. Combining the time-independence of both the squared velocity  $\dot{\vec{x}}^2$  and the energy density we obtain the formula

$$\mathcal{P}_0(\sigma^r) = T \sqrt{\frac{|g|}{1 - \vec{v}^2(\sigma^r)}}. \quad (39)$$

which shows time-independence of the brane volume, i.e.  $\dot{g} = 0$ . These conditions are characteristic of spinning  $p$ -branes with their elastic force compensated by the centrifugal force. This proves that the solutions of the equations  $\Delta^{(p)}\vec{x} = 0$  must describe spinning  $p$ -branes. To find such solutions in explicit form we restrict ourselves by the case when spinning  $p$ -branes evolve in odd-dimensional Minkowski space with the fixed dimension  $D = (2p + 1)$ .

In this case we have  $p$  independent components of  $\vec{x}(t, \sigma^r)$  remaining after the solution of the  $p$  orthogonality constraints  $(\dot{\vec{x}} \cdot \partial_r \vec{x}) = 0$ . In view of the above-derived  $p$  Noether identities we have just  $p(= 2p - p)$  independent equations for  $p$  remaining degrees of freedom of  $\vec{x}(t, \sigma^r)$ . In addition there are  $p$   $\sigma$ -dependent diffeomorphisms (10) which can be used to fix  $\sigma$ -dependence of these DOF. Finally, the brane equations are reduced to the system of  $p$  usual differential equations for  $p$  functions independent of  $\sigma^r$ . A possible way to accomplish such a type of reduction is, e.g. to separate  $t$  and  $\sigma$  variables in each component of the vector  $\vec{x}(t, \sigma^r)$

$$x_i(t, \sigma^r) = u_i(t)v_i(\sigma^r) \quad (40)$$

with subsequent exclusion of gauge and non-propagating DOF using  $p$  orthogonality conditions (4) and  $p$  additional gauge conditions for the remaining diffeomorphisms (10). This strategy was realized in [25], where the discussed  $2p$ -dimensional Euclidean vector  $\vec{x}(t, \sigma^r)$  of spinning  $p$ -brane was presented as the generalization of the membrane anzatses studied in [21] and [22]

$$\begin{aligned} \vec{x}^T(t, \sigma^r) &= (q_1 \cos \theta_1, q_1 \sin \theta_1, q_2 \cos \theta_2, q_2 \sin \theta_2, \dots, q_p \cos \theta_p, q_p \sin \theta_p), \quad (41) \\ q_a &= q_a(\sigma^r), \quad \theta_a = \theta_a(t) \end{aligned}$$

which gives a solution of orthogonality constraints (4) with the propagating DOFs represented by the polar angles  $\theta_a(t)$ . This anzats gives

$$\dot{\vec{x}}^2 = \sum_{a=1}^p q_a^2 \dot{\theta}_a^2.$$

Keeping in mind constraint (38) we obtain the following solution for  $\theta_a(t)$

$$\sum_{a=1}^p q_a^2 \dot{\theta}_a^2(t) = \dot{v}^2(\sigma^r) \longrightarrow \theta_a(t) = \theta_{a0} + \omega_a t, \quad (42)$$

where  $\theta_{0a}$  and  $\omega_a$  are the integration constants with  $a=1,2,\dots,p$ .

As a result, the energy density of spinning  $p$ -brane  $\mathcal{P}_0$  (40) is defined by the following function of the velocity components  $\omega_a q_a(\sigma^r)$

$$\mathcal{P}_0 = T \sqrt{\frac{|g|}{1 - \sum_{a=1}^p \omega_a^2 q_a^2}}. \quad (43)$$

This time-independent energy density turns into the density  $\mathcal{P}_0^{(stat)}$  of a static brane in the limiting case of all the vanishing frequencies:  $\omega_a = 0$ .

The separation between  $t$  and  $\sigma^r$  variables realized by anzats (41) turns out to be a sufficient condition for exact solvability of Eqs.(41). Indeed, the substitution of (41) into (16) reduces these  $2p$  nonlinear PDEs for the components of  $\vec{x}$  to  $p$  PDEs for the  $p$  components of  $\mathbf{q}(\sigma^r) := (q_1, \dots, q_p)$ .

$$-\omega_a^2 q_a + \sum_{b,r,s=1}^p \omega_b^2 q_b (q_{b,r} g^{rs} q_{a,s}) = (1 - \sum_{b=1}^p q_b^2 \omega_b^2) \Delta^{(p)} q_a, \quad (44)$$

Because  $\Delta^{(p)} q_a = 0$ , as a consequence of  $\Delta^{(p)} x_m = 0$ , Eqs. (44) are satisfied if there is exact cancellation between all its terms. The cancellation occurs when the conditions

$$g^{rs} q_{a,r} q_{b,s} = \delta_{ab}, \quad g_{rs} = q_{a,r} q_{a,s} \quad (45)$$

for the induced metric  $g_{rs}$  on  $\Sigma_p$  generated by (41) are satisfied. These conditions express the space-like part of metric (4) exactly in the form connecting its with the components of the  $p$ -bein  $e_r^a$  attached to the hypersurface  $\Sigma_p$ .

As a result, the partial derivatives  $q_{,r}^a$  coincide with the  $p$ -bein  $e_r^a$  and conditions (45) may be presented in the equivalent form as

$$e_r^a = q_{,r}^a. \quad (46)$$

The worldvolume metric  $G_{\alpha\beta}$  on  $\Sigma_{p+1}$  generated by anzats (41) is given by

$$G_{tt} = 1 - \sum_{a=1}^p q_a^2 \omega_a^2, \quad g_{rs} = \sum_{a=1}^p q_{a,r} q_{a,s} \equiv \mathbf{q}_{,r} \mathbf{q}_{,s}, \quad \mathbf{q} := (q_1, \dots, q_p) \quad (47)$$

which yields the following squared interval  $ds_{p+1}^2$  on  $\Sigma_{p+1}$

$$ds_{p+1}^2 = (1 - \sum_{a=1}^p q_a^2 \omega_a^2) dt^2 - \sum_{a=1}^p dq_a dq_a. \quad (48)$$

This shows that in terms of the new coordinates  $q_a(\sigma^r)$  the hypersurface  $\Sigma_p$  metric  $g_{rs}$  becomes independent of  $\sigma^r$ .

For infinite  $p$ -branes without boundary conditions and  $-\infty < \sigma^r < +\infty$  one can choose the following gauge for the residual symmetry (10)

$$q_1(\sigma^r) = k\sigma^1, \quad q_2(\sigma^r) = k\sigma^2, \quad \dots, \quad q_p(\sigma^r) = k\sigma^p, \quad (49)$$

where  $k \sim T^{\frac{-1}{p+1}}$  is an arbitrary constant with the dimension of length. This choice results in the constant diagonal matrices for  $p$ -bein  $e_r^a$  and metric  $g_{rs}$

$$e_r^a = k\delta_r^a, \quad g_{rs} = k^2\delta_{rs} \quad (50)$$

which solve the considered harmonic equations  $\Delta^{(p)} q_a(\sigma^r) = \Delta^{(p)} \vec{x}(t, \sigma^r) = 0$ .

It proves that the initial value constraints  $\Delta^{(p)} \vec{x} = 0$  select exact solutions of Eqs.(22) describing spinning branes with the shape of  $p$ -dim. hyperplanes

$$\begin{aligned} \vec{x}^T(t, \sigma^r) = & k(\sigma^1 \cos(\theta_{10} + \omega_1 t), \sigma^1 \sin(\theta_{10} + \omega_1 t), \dots, \\ & \sigma^p \cos(\theta_{p0} + \omega_p t), \sigma^p \sin(\theta_{p0} + \omega_p t)) \end{aligned} \quad (51)$$

The energy density of the infinite spinning branes is given by

$$\mathcal{P}_0(\sigma^r) = \frac{Tk^p}{\sqrt{1 - k^2 \sum_{a=1}^p \sigma_a^2 \omega_a^2}} \quad (52)$$

and one can see that the condition  $k\omega \sim \omega T^{\frac{-1}{p+1}} \rightarrow 0$  has to be satisfied when  $|\sigma_a| \rightarrow \infty$  to preserve the real value of  $\mathcal{P}_0$ . This demands  $\omega \rightarrow 0$

when the tension  $T$  is fixed, and thus  $\mathcal{P}_0$  (51) becomes a constant  $\sim T^{\frac{1}{p+1}}$  resulting in the divergent total energy in the static limit because of the infinite integration range in the parameters  $\sigma_a$ . The static solutions may be treated as domain "hyperwalls" generalizing the well-known two-dimensional domain walls which appear as solutions in various physical models.

The integration range in  $\sigma$  can be made a compact by considering closed or open branes with the corresponding boundary conditons. Below we consider the case of closed spinning  $p$ -branes described by the anzats (41).

## 5 Folded $p$ -branes as solutions of $\Delta^{(p)}\vec{x} = 0$

The change of gauge conditions (49) into the ones considered in [25]

$$q_1(\sigma^r) = q_1(\sigma^1), \quad q_2(\sigma^r) = q_2(\sigma^2), \quad \dots, \quad q_p(\sigma^r) = q_p(\sigma^p), \quad (53)$$

where each of the functions  $q_a$  is a monotonic continuous function of only the variable  $\sigma^r$  with  $r = a$ , gives more general solutions for conditions (45)

$$q_{a,r} = \delta_{as} \dot{q}_r, \quad \dot{q}_s := \frac{dq_s}{d\sigma^s},$$

$$g_{rs} = \delta_{rs} \dot{q}_s^2, \quad g^{rs} = \frac{\delta_{rs}}{\dot{q}_s^2}, \quad g = \prod_{a=1}^p \dot{q}_a^2 \equiv \prod \dot{q}_a^2 \quad (54)$$

with the diagonal matrices  $q_{a,r}$  and  $g_{rs}$ , and factorized determinant of  $g_{rs}$ . The radial components  $q_a(\sigma^r)$  (53) and metric (54) are the solutions of eqs.  $\Delta^{(p)}\vec{x} = 0$ . To verify the statement it is enough to prove that these  $q$ -coordinates are the solutions of the reduced harmonic equations

$$\Delta^{(p)} q_a(\sigma^r) = 0, \quad (a = 1, 2, \dots, p). \quad (55)$$

This becomes evident after the substitution of (54) into (55) resulting in

$$\Delta^{(p)} q_a = \frac{1}{\prod \dot{q}_b} \frac{\partial}{\partial \sigma^a} \left( \frac{\prod \dot{q}_b}{\dot{q}_a} \right) = 0. \quad (56)$$

The latter equations are satisfied in view of cancellation of the derivative  $\dot{q}_a$  which is only one function depending on  $\sigma^a$  in the fraction  $\frac{\prod \dot{q}_b}{\dot{q}_a}$ .

It is clear, that the mapping (53) with regular monotonic  $q$ -functions describes the same infinite  $p$ -dimensional hyperplanes as the solution (49).

However, the replacement of the monotonic  $q$ -functions by the periodic ones with isolated nonregular points in  $g^{rs}$  (54) yields solutions of Eqs.  $\Delta^{(p)}\vec{x} = 0$  describing compact folded  $p$ -branes. The solutions generalize ones describing the folded strings [26], [28], [27] to the case of  $p$ -branes. The folds arise as a result of the one-parametric dependence of the functions  $q_a(\sigma^a)$  (53) applied to describe closed  $p$ -brane by the generalized anzats (51)

$$\vec{x}^T(t, \sigma^r) = (q_1(\sigma^1) \cos(\theta_{10} + \omega_1 t), q_1(\sigma^1) \sin(\theta_{10} + \omega_1 t), \dots, q_p(\sigma^p) \cos(\theta_{p0} + \omega_p t), q_p(\sigma^p) \sin(\theta_{p0} + \omega_p t)) \quad (57)$$

with the initial data  $\theta_{0a} = 0$  at  $t = 0$  and the density energy (43) given by

$$\mathcal{P}_0 = \frac{T |\prod \dot{q}_a|}{\sqrt{1 - \sum_{a=1}^p q_a^2 \omega_a^2}}. \quad (58)$$

In the case of closed  $p$ -branes their  $\sigma$ -parameters are bounded:  $\sigma^r \in [0, 2\pi]$ , and therefore each of the functions  $q_a(\sigma^r)$  from (57) has to be a periodic one:  $q_a(0) = q_a(2\pi)$ . Next we see that at any moment  $t$  the world vector  $\vec{x}^T(t, \sigma^r)$  (57) is produced from  $\vec{x}_0^T(\sigma^r) = (q_1, 0, q_2, 0, \dots, q_p, 0)$  by the time-parametrized rotations belonging to the diagonal subgroup  $U(1)^p$  of the group  $SO(2p)$ . This subgroup is composed of the time-dependent rotations in the planes  $x_1 x_2, x_3 x_4, \dots, x_{2p-1} x_{2p}$  about the angles  $\theta_a = \theta_{0a} + \omega_a t$ , respectively. Thus, the  $p$ -brane worldvolume is formed by the rotations of the closed  $p$ -brane initially embedded into the  $p$ -dim. subspace spanned by all odd coordinate axes of the considered  $2p$ -dim. Euclidean space. These rotations preserve the initial brane shape. So, the periodicity conditions for  $q_1$  with respect to  $\sigma_1$ ,  $q_2$  with respect to  $\sigma_2$ , etc. will be satisfied if the  $p$ -brane is initially folded up along each of the odd coordinate axes. A simple example of the solution is given by the symmetrically folded closed  $p$ -brane

$$\vec{x}^T(0, \sigma^r) = k(|\pi - \sigma^1|, 0, |\pi - \sigma^2|, 0, |\pi - \sigma^2|, \dots, |\pi - \sigma^p|, 0) \quad (59)$$

with the functions  $q_a(\sigma^a) = k|\pi - \sigma^a|$  which realize the conditions  $q_a(0) = q_a(2\pi)$  by the bending formation at  $\sigma^a = \pi$  which create additional forces orthogonal to  $\Sigma_p$  around these points. The latters fix the lines (planes) on the brane hypersurface  $\Sigma_p$  along which it is bent. For the folded membrane ( $p = 2$ ) embedded into 4-dim. Euclidean space its image may be visualized as a double-folded sheet of paper forming a stack of four equal small squares originated from the original unfolded square with the side length equal to

$2k\pi$ . The functions  $q_a(\sigma^a)$  in (59) are continuous ones, but their derivatives have the jump discontinuity equal to  $2 = 1 - (-1)$  at  $\sigma^a = \pi$ . These jumps result in the indefiniteness of the induced metric (54) at these points. The change of the parametrization (59) by

$$\vec{x}^T(0, \sigma^r) = k(\sin \frac{\sigma^1}{2}, 0, \sin \frac{\sigma^2}{2}, 0, \dots, \sin \frac{\sigma^p}{2}, 0) \quad (60)$$

smooths out the derivative jumps at  $\sigma^a = \pi$ . The flat metric  $g_{rs}$  (54) vanish at these points, as well as the energy density  $\mathcal{P}_0$  (58) (if  $\sum_{a=1}^p q_a^2 \omega_a^2 \neq 1$ ).

A more general parametrization producing  $(n_1, n_2, \dots, n_p)$  singular points for  $g^{rs}$  defined by the functions  $(q_1, q_2, \dots, q_p)$  (54), respectively, may be chosen in the form similar to the one considered in [26]

$$\vec{x}^T(0, \sigma^r) = k(\sin \frac{n_1 \sigma^1}{2}, 0, \sin \frac{n_2 \sigma^2}{2}, 0, \dots, \sin \frac{n_p \sigma^p}{2}, 0) \quad (61)$$

with the set  $(n_1, n_2, \dots, n_p)$  treated as the topological winding numbers.

So, ansatz (57) with the periodic  $q$ -functions gives exact solutions of  $\Delta^{(p)} \vec{x} = 0$  with isolated singularities in  $g^{rs}$  and describe initially folded branes. The brane worldvolume  $\Sigma_{p+1}$  associated with the initially folded hypersurface  $\Sigma_p$  is produced by its rotations as a whole realized by the above mentioned Abelian group  $U(1) \times U(1) \times \dots \times U(1) \equiv U(1)^p$ . The corresponding rotation angles  $\theta_a$  are treated as the generalized cyclic coordinates of the Hamiltonian density (5) corresponding to the energy density  $\mathcal{P}_0$  (43). The momenta  $j_a$  conjugate to the generalized coordinates  $\theta_a$

$$j_a := \frac{\partial \mathcal{L}}{\partial \dot{\theta}_a} = \vec{\mathcal{P}} \frac{\partial \vec{x}}{\partial \dot{\theta}_a}$$

are given by

$$j_a = \mathcal{P}_0 q_a^2 \dot{\theta}_a \equiv \mathcal{P}_0 \omega_a q_a^2. \quad (62)$$

Then the corresponding Hamiltonian  $p$ -brane density takes the form

$$\mathcal{H}_0 = \sqrt{\sum_{a=1}^p (j_a/q_a)^2 + T^2 |g|}. \quad (63)$$

The momenta (62) are integrals of the motion

$$\frac{dj_a}{dt} = 0, \quad (a = 1, 2, \dots, p)$$

proportional to the conserved energy density  $\mathcal{P}_0$ . The values  $j_a$  are the components of the angular momentum density associated with the generators of rotations in the planes  $x_1x_2, x_3x_4, \dots, x_{2p-1}x_{2p}$  which form the above-discussed Abelian group  $U(1)^p$ . They may be presented as explicit functions of the non-propagating brane coordinates  $q_a(\sigma^r)$  and their derivatives

$$j_b = T\omega_b q_b^2 \sqrt{\frac{|g|}{1 - \sum_{a=1}^p \omega_a^2 q_a^2}}. \quad (64)$$

We conclude that the choice of the initial value constraints in the form of harmonicity conditions (33) selects the regular or singular  $g^{rs}$  given by the solutions of Eqs.(22) describing infinite or compact folded spinning  $p$ -branes.

## 6 Solvable $p$ -brane motions with $\Delta^{(p)}\vec{x} = -\Lambda\vec{x}$

In the previous section we have found that the harmonicity equations  $\Delta^{(p)}\vec{x} = 0$  treated as the initial value constraints provide the exact solutions [25] of brane equations. One can conjecture that specially constructed deformations of the harmonicity conditions may reveal other exact solutions. This proposal is compatible with the specific form of brane Eqs.(22), where the shift of the factor  $G_{tt}$  to their l.h.s. leaves only  $\Delta^{(p)}\vec{x}$  in the r.h.s.. Therefore, the time derivatives of  $\vec{x}$  are concentrated in the l.h.s. of (22). Using various initial value constraints, including  $\Delta^{(p)}\vec{x}$  in combination with  $\vec{x}$  and  $\dot{\vec{x}}$ , one can generate various evolution equations. It may occur that some of these evolution equations are exactly solvable like in the case  $\Delta^{(p)}\vec{x} = 0$ . The constraint deformations are under control of the Noether identities demanding  $\Delta^{(p)}\vec{x}$  to be an eigenvector of the projection operator  $\Pi_{ik}$ , as it follows from (24). Variation of the constraints will result in deformations of the brane shape selfconsistent with the evolution equations.

As an example realizing this proposal and generalizing the solutions [24] we consider the following invariant deformation of the harmonicity conditions

$$\Delta^{(p)}\vec{x} = -\Lambda\vec{x}, \quad (65)$$

where  $\Lambda$  is an arbitrary function invariant under diffeomorphisms of the hypersurface  $\Sigma_p$ . The substitution of (65) into (22) yields the evolution equation

$$\ddot{\vec{x}} - (\ddot{\vec{x}}\vec{x}^s)\vec{x}_{,s} = -\Lambda(1 - \dot{\vec{x}}^2)\vec{x}, \quad (66)$$

$$(\partial_s \ln \sqrt{|g|})\vec{x}^s + \partial_s \vec{x}^s = -\Lambda\vec{x} \quad (67)$$

accompanied with the constraints (67) for the initial value for this evolution equation. Due to the Noether identities we obtain that the projections of Eqs.(66-67) on  $\vec{x}_r$  result in the following rotationally invariant constraint

$$\vec{x}^2(t, \sigma^r) = \vec{R}^2(t) \quad (68)$$

which shows that the  $p$ -brane hypersurface  $\Sigma_p$  resides on the  $(D-2)$ -dimensional sphere of the radius  $R = \sqrt{\vec{R}^2(t)}$  embedded into the  $(D-1)$ -dimensional Euclidean space. The projections of (66-67) on  $\vec{x}$  yield the equations

$$\frac{1}{2} \frac{d^2 \vec{x}^2}{dt^2} = 1 - (p+1)(1 - \dot{\vec{x}}^2), \quad (69)$$

$$\Lambda(t, \sigma^r) = \frac{p}{\vec{R}^2(t)} \quad (70)$$

fixing the unknown function  $\Lambda$ . The projections of (66-67) on  $\vec{x}$  give the relation

$$1 - \dot{\vec{x}}^2 = \left( \frac{\vec{R}^2(t)}{l^2} \right)^p, \quad (71)$$

where  $l$  is the integration constant with the dimension  $[l] = L$ . The latter relation in combination with (69) yields the closed equation for  $\xi := \vec{R}^2(t)$

$$\frac{1}{2} \ddot{\xi} = 1 - (p+1) \left( \frac{\xi}{l^2} \right)^p. \quad (72)$$

The first integral of Eq. (72) is given by the relation

$$l^2 \dot{\zeta}^2 = (1 - \zeta^p)(1 + \zeta^p)$$

expressed in terms of the new dimensionless variable  $\zeta(t) := \frac{\sqrt{\xi}}{l} \equiv \frac{\sqrt{\vec{R}^2}}{l}$  substituted instead of  $\vec{R}^2(t)$ . Then the first integral is presented as

$$\left( \frac{d\zeta}{d\eta} \right)^2 = \frac{1}{2} (1 - \zeta^p)(1 + \zeta^p) \quad (73)$$

after the transition to the new rescaled time variable  $\eta := \frac{\sqrt{2}t}{l}$ . For the case  $p = 2$  corresponding to membrane Eq. (73) is the defining equation for the



Jacobi elliptic cosine  $cn(\eta, k)$  with the elliptic modulus  $k = \frac{1}{\sqrt{2}}$ . For  $p > 2$  the exact solution of (73) is given by the hyperelliptic integral

$$\eta = \pm \sqrt{2} \int \frac{d\zeta}{\sqrt{1 - \zeta^{2p}}} + \text{const} \quad (74)$$

generalizing the elliptic membrane solution to  $p$ -branes with arbitrary  $p$ . Thus, we obtain exact solution for the length  $\sqrt{\vec{R}^2(t)}$  of  $\vec{x}$  without any gauge fixing for the symmetry (10) and the restriction  $D = 2p + 1$  [24].

Then the generalized harmonicity conditions (65) take the form

$$\Delta^{(p)} \vec{x} = -\frac{p}{\vec{R}^2(t)} \vec{x} \quad (75)$$

with the known function  $\vec{R}^2(t)$  depending only on time. The  $\sigma$ -independence of  $\vec{x}^2 = \vec{R}^2(t)$  results in the  $\sigma^r$ -independence of  $\dot{\vec{x}}^2$ , as it follows from (71) and the fact that the second term in the l.h.s. of (66) vanishes. As a result, Eqs. (66) and (67) are reduced to two connected subsystems

$$\ddot{\vec{x}} + \frac{p}{l^2} \left( \frac{\vec{R}^2}{l^2} \right)^{p-1} \vec{x} = 0, \quad (76)$$

$$\Delta^{(p)} \vec{x} + \frac{p}{\vec{R}^2} \vec{x} = 0 \quad (77)$$

with the evolution equations describing  $2p$ -dim. oscillator with time-dependent frequency given by the (hyper)elliptic function of time. To find all the components of the vector  $\vec{x}$  we must solve Eqs.(76) and (77). Since the length of  $\vec{x}$  is  $\sigma$ -independent, this dependence concentrates in the direction cosines of  $\vec{x}$ . This suggests representation of  $\vec{x}$  in the form  $x_i(t, \sigma^r) = \mathcal{O}_{ik}(t, \sigma^r) R_k(t)$ , where  $\mathcal{O}_{ik} \in SO(D - 1)$  group of rotations of  $(D - 1)$ -dimensional subspace of the Minkowski space. In view of the time independence of  $|\vec{x}|$ , the time derivative of this representation for  $\vec{x}$  shows that the matrix  $\mathcal{O}$  is also time-independent. This observation results in the separation of variables

$$x_i(t, \sigma^r) = \mathcal{O}_{ik}(\sigma^r) R_k(t), \quad \mathcal{O}_{ik} \mathcal{O}_{jk} = \delta_{ij}. \quad (78)$$

Similarly to the spinning brane case we restrict ourselves by  $(2p + 1)$ -dim. Minkowski space and choose the matrix  $\mathcal{O}_{ik}$  from the Abelian subgroup  $O(2)^p$  of the group  $SO(2p)$ . Then  $\vec{x}$  takes the form of the anzats [24]

$$\begin{aligned} \vec{x}^T &= (q_1 \cos \theta_1, q_1 \sin \theta_1, q_2 \cos \theta_2, q_2 \sin \theta_2, \dots, q_p \cos \theta_p, q_p \sin \theta_p), \\ q_a &= q_a(t), \quad \theta_a = \theta_a(\sigma^r). \end{aligned} \quad (79)$$

Contrary to the spinning anzats (41), considering its polar angles to be propagating DOF, here we have the radial coordinates  $\mathbf{q}(t) = (q_1, \dots, q_p)$  as the propagating DOF. Anzats (79) yields the following expressions for the lengths of  $\vec{x}$  and  $\dot{\vec{x}}$

$$\vec{x}^2(t, \sigma^r) = \mathbf{q}^2(t) \equiv \sum_{a=1}^p q_a^2(t), \quad \dot{\vec{x}}^2(t) = \dot{\mathbf{q}}^2(t) \quad (80)$$

and for the worldvolume metric  $G_{\alpha\beta}$  on  $\Sigma_{p+1}$ , respectively

$$G_{tt} = 1 - \dot{\mathbf{q}}^2, \quad \mathbf{q} := (q_1, \dots, q_p), \quad g_{rs} = \sum_{a=1}^p q_a^2 \theta_{a,r} \theta_{a,s}, \quad (81)$$

where  $\theta_{a,r} \equiv \partial_r \theta_a$ . The corresponding squared interval  $ds_{p+1}^2$  is given by

$$ds_{p+1}^2 = (1 - \dot{\mathbf{q}}^2) dt^2 - \sum_{a=1}^p q_a^2(t) d\theta_a d\theta_a. \quad (82)$$

Representation (82) shows that in the new coordinates  $\theta_a(\sigma^r)$ , used instead of  $\sigma^r$ , the metric on  $\Sigma_p$  becomes independent of  $\sigma^r$  with  $p$  Killing vector fields represented by the derivatives  $\frac{\partial}{\partial \theta_a}$ . Thus, anzats (79) describes  $p$ -dimensional torus  $S^1 \times S^1 \times \dots \times S^1$  with zero curvature and the time-dependent radii  $q_a$ . This anzats reduces the number of degrees of freedom to  $p$  carried by the radial coordinates  $q_a$  which obey reduced Eqs.(76)

$$\ddot{\mathbf{q}} = -\frac{p}{l^2} \left( \frac{\mathbf{q}^2}{l^2} \right)^{p-1} \mathbf{q} \quad (83)$$

with their first integral equal to  $1 - \dot{\mathbf{q}}^2 = \left( \frac{\mathbf{q}^2(t)}{l^2} \right)^p$ . The substitution of expressions (80) in Eqs.(69) regenerate Eq. (73) and its hyperelliptic solution (74) with  $\mathbf{q}^2$  substituted for  $\vec{x}^2$ , e.d.  $\eta(t) = \frac{\sqrt{\mathbf{q}^2}}{l}$ .

The substitution of anzats (79) into Eqs. (77) transforms them to homogeneous equations for the components  $\theta_a$  which are equivalent to

$$g^{rs} \theta_{a,r} \theta_{a,s} = \frac{p}{\mathbf{q}^2} \quad (a = 1, 2, \dots, p), \quad (84)$$

$$\frac{1}{\sqrt{|g|}} \partial_r \left( \sqrt{|g|} g^{rs} \theta_{a,s} \right) + g^{rs} \theta_{a,rs} = 0 \quad (85)$$

for each  $a$ . The equations are easily solved in the gauge  $\theta_a = \delta_{ar} \sigma^r$  [24]

$$\theta_1(\sigma^r) = \sigma^1, \quad \theta_2(\sigma^r) = \sigma^2, \quad \dots, \quad \theta_p(\sigma^r) = \sigma^p, \quad (86)$$

where  $\sigma^r$ -independent metric  $g_{rs}(t)$  takes the following diagonal form

$$g_{rs}(t) = q_r^2(t)\delta_{rs}, \quad g = (q_1 q_2 \dots q_p)^2 \quad (87)$$

and transforms Eqs. (85) to identities. Eqs. (84) reduce to the conditions

$$\mathbf{q}^2 \equiv \sum_{a=1}^p q_a^2 = p q_1^2 = p q_2^2 = \dots = p q_p^2 \quad (88)$$

which mean coincidence of all  $q_a$ -functions:  $q_a(t) \equiv q(t)$ .

From the geometrical point of view the coincidence condition picks up the case of degenerate  $p$ -torus with equal radii [24]. In view of the above constraints, the system of  $p$  tangled equations (83)

$$\ddot{q}_a = -\frac{p}{l^2} \left( \frac{q_1^2 + q_2^2 + \dots + q_p^2}{l^2} \right)^{p-1} q_a \quad (a = 1, 2, \dots, p) \quad (89)$$

shrinks to the single exactly solvable nonlinear equation

$$\ddot{q} + \frac{p}{l^2} \left( \frac{p q^2}{l^2} \right)^{p-1} q = 0 \quad (90)$$

with the above-studied first integral given by

$$1 - p \dot{q}^2 = \left( \frac{p q^2}{l^2} \right)^p. \quad (91)$$

The change of variables  $\tilde{\zeta} = \frac{\sqrt{p} q}{l}$ ,  $\eta = \frac{\sqrt{2} t}{l}$  transforms Eq. (91) into Eq. (73)

$$\left( \frac{d\tilde{\zeta}}{d\eta} \right)^2 = \frac{1}{2} (1 - \tilde{\zeta}^p)(1 + \tilde{\zeta}^p) \quad (92)$$

and its solution is given by the considered hyperelliptic integral (74)

$$\eta = \pm \sqrt{2} \int \frac{d\tilde{\zeta}}{\sqrt{1 - \tilde{\zeta}^{2p}}} + const. \quad (93)$$

Thus, we proved that the deformation (65) of the harmonicity conditions selects the exact solution which describes collapsing  $p$ -brane with the shape of the degenerate  $p$ -torus [24].

## 7 Summary

A new approach to the problem of exact solvability of nonlinear  $p$ -brane equations and constraints in  $D$ -dimensional Minkowski space was considered. The approach is based on the connection between the initial value problem for the brane equations and their exact solutions.

The  $p$ -brane equations, initially written in the form of  $(p+1)$ -dimensional worldvolume wave equations, were reduced in the orthogonal gauge to  $p$ -dimensional equations with their r.h.s. presented by  $\Delta^{(p)}\vec{x}$  and l.h.s. equal to the brane acceleration projection on the directions orthogonal to its hypersurface  $\Sigma_p$ . The Noether identities associated with the diffeomorphisms of the brane worldvolume  $\Sigma_{p+1}$  were derived and used for the choice of the admissible constraints for the initial data. Two types of such constraints were studied and the corresponding exact solutions were obtained. The first of them considers the harmonicity constraints  $\Delta^{(p)}\vec{x} = 0$  which select spinning  $p$ -branes. In the case  $D = 2p + 1$  the harmonicity constraints are exactly solved by the ansatz previously considered in [25]. These solutions include either regular solutions for  $g^{rs}$  describing infinite  $p$ -branes with the shape of  $p$ -dimensional hyperplanes or nonregular  $g^{rs}$  associated with folded compact  $p$ -branes. The case of the infinite branes includes static  $p$ -branes with the constant density of energy treated as  $p$ -dimensional domain walls. The second set is picked up by the deformed harmonicity conditions  $\Delta^{(p)} = -\Lambda\vec{x}$  and describes closed  $p$ -brane lying on a collapsing sphere  $S^{D-2}$  embedded into  $(D-1)$ -dimensional Euclidean subspace of  $D$ -dimensional Minkowski space with arbitrary  $D > 4$ . The time-dependent radius of the sphere is presented by hyperelliptic functions. In the particular case of odd  $D = 2p + 1$  the  $p$ -brane hypersurface  $\Sigma_p$  turns out to be isometric to flat collapsing  $p$ -dimensional torus which coincides with the exact solution [24]. The described spinning or collapsing 5-branes ( $p = 5$ ) give exact solutions of  $D = 11$  M/string theory and it is interesting to understand the physics associated with them.

Extension of the proposed approach to the case of opened  $p$ -branes with various boundary conditions as well as its generalization to the case of known cosmological backgrounds seems to be interesting.

### Acknowledgments

The author is grateful to Physics Department of Stockholm University and Nordic Institute for Theoretical NORDITA for kind hospitality and financial support. The results were presented at the Conference STDE-2012 in honor of Vladimir Aleksandrovich Marchenko's 90th birthday [29].

## References

- [1] P. K. Townsend, *The Eleven-dimensional Supermembrane Revisited*, Phys. Lett. B 350 (1995) 184; E. Witten, *String Theory Dynamics In Various Dimensions*, Nucl. Phys. B 443 (1995) 85; T. Banks, W. Fishler, S.H. Shenker and L. Susskind, *M theory as a Matrix Mode: A conjecture* Phys. Rev. D 55(8):5112; M. J. Duff, *The world in Eleven Dimensions: Supergravity, Supermembranes and M-theory* (IOP, Bristol 1999).
- [2] R. W. Tucker, *Extended Particles and the Exterior Calculus*, Lectures given at the Rutherford Laboratory, February 1976.
- [3] J. Hoppe, MIT PhD Thesis, 1982, and in Proc. Int. Workshop on Constraints Theory and Relativistic Dynamics. G. Longhi, L. Lusanna (eds), Singapore: World Scientific 1987.
- [4] E. Bergshoeff, E. Sezgin and P.K. Townsend, Phys. Lett. B 189 (1987) 75.
- [5] M. Duff, P. Howe, T. Inami and K. Stelle, Phys. Lett. B 191 (1987) 70.
- [6] E. Floratos and J. Illipoulos, Phys. Lett. B 201 (1988) 237.
- [7] B. de Witt, J. Hoppe and G. Nicolai, Nucl. Phys. B 305 [FS23] (1988) 545.
- [8] A.A. Zheltukhin, Sov. J. Nucl. Phys. 48 (1988) 375.
- [9] B. de Witt, M. Lusher and G. Nicolai, Nucl. Phys. B 320 (1989) 135.
- [10] I.A. Bandos and A.A. Zheltukhin, Fortsch. Phys. 41 (1993) 619.
- [11] M. Bordemann, J. Hoppe, Phys. Lett. B 317 (1993) 315; ibid. B 325 (1994) 359.
- [12] J. Hoppe, Phys. Lett. B 329 (1994)
- [13] J. Polchinski, Phys. Rev. Lett. 75 (1995) 4729.
- [14] D.V. Uvarov and A.A. Zheltukhin, Phys. Lett. B 545 (2002) 183.
- [15] P.A. Collins and R. W. Tucker, Nucl. Phys. B 112 (1976) 150.

- [16] K. Kikkawa and M. Yamasaki, Progr. Theor. Phys. 76 (1986) 1379.
- [17] J. Hoppe and H. Nicolai, Phys. Lett. B 196 (1987) 451.
- [18] M. Axenidis, E.G. Floratos and L. Perivolaropoulos, Phys. Rev. D 66 (2002) 085006.
- [19] J. Arnlind, J. Hoppe and S. Theisen, Phys. Lett. B 599 (2004) 118.
- [20] M. Axenides and E. G. Floratos , JHEP 0703 (2007) 093.
- [21] J. Hoppe, Compl. Anal. Oper. Theory 3 (2009) 419.
- [22] M. Trzetrzelewski and A.A. Zheltukhin, Phys. Lett. B 679 (2009) 523.
- [23] A.A. Zheltukhin and M.Trzetrzelewski, J. Math. Phys. 51 (2010) 062303.
- [24] A.A. Zheltukhin, Nucl. Phys. B 858 (2012) 142;  
Prob.Atomic Sci.Technol. 2012 N1 (2012) 7-10.
- [25] A.A. Zheltukhin, Int. J. Geom. Meth. Mod. Phys. 09 (2012) 1261017.
- [26] A.A. Zheltukhin, Sov. J. Nucl. Phys. 34 (1981) 562; Lett. in Math. Phys. 5 (1981) 213; Phys. Lett. B 116 (1982) 147.
- [27] S.V. Klimenko, I.N. Nikitin, Il Nuovo Cimento A, 111 (1998) 1431; L. Nikitina, I. Nikitin and S. Klimenko, Electronic journal "INVESTIGATED IN RUSSIA", <http://zhurnal.ape.relarn.ru/articles/2003/037e.pdf>.
- [28] J. Hoppe, "Conservation laws and formation of singularities in relativistic theories of extended objects theories" [arXiv:hep-th/9503069]
- [29] A.A. Zheltukhin, *On Some Nonlinear Equations In (Mem)brane Dynamics*, Book of Abstracts of International Conference "Spectral Theory and Differential Equations" in Honor of Vladimir A. Marchenko's 90th birthday: STDE-2012, August 20-24, 2012, Kharkiv, Ukraine, p. 118.